UDC 551.501.45:551.577.2:551.509.617

Three-Parameter Kappa Distribution Maximum Likelihood Estimates and Likelihood Ratio Tests

PAUL W. MIELKE, JR.—*Department of Statistics, Colorado State University, Fort Collins, Colo.*

EARL S. JOHNSON—*Department of Statistics, Mary Washington College, Fredericksburg, Va.*

ABSTRACT—Methods are presented for obtaining maximum likelihood estimates and tests of hypotheses involving the three-parameter kappa distribution. The obtained methods are then applied by fitting this distribution to realized sets of precipitation and streamflow data and testing for seeding effect differences between realized seeded and nonseeded sets of precipitation data. The kappa distribution appears to fit precipitation data as well as either the gamma or log-normal distribution. As a consequence, the sensitivity of test procedures based on the kappa distribution compares favorably with that of previously used test procedures.

Since both the density and cumulative distribution functions of the kappa distribution are in closed form, the density and cumulative distribution functions associated with each order statistic are also in closed form. In contrast, the gamma and log-normal cumulative distribution functions are not in closed form. As a consequence, computations involving order statistics are far more convenient with the kappa distribution than either the gamma or log-normal distributions.

1. INTRODUCTION

The gamma and log-normal families of distributions (or simply gamma and log-normal distributions) are commonly used to fit precipitation data in meteorological applications. Each of these distributions is characterized by two parameters termed the "shape" and "scale" parameters. In weather modification applications, attention is focused on detecting scale changes in precipitation amounts induced by cloud seeding (Thom 1957, Neyman and Scott 1965, Simpson 1972).

Recently another family of probability distribution functions was introduced that is relatively easy to work with, possesses a simple closed form for both its density and cumulative distribution functions, and appears to be appropriate in applications involving precipitation data (Mielke 1973). This family has been termed the "three-parameter kappa family of distributions" [or simply the "kappa(3) distribution"]. The kappa(3) distribution's density function and cumulative distribution function are given by

$$f(x)=\begin{cases}\dfrac{\alpha\theta}{\beta}\left(\dfrac{x}{\beta}\right)^{\theta-1}\left[\alpha+\left(\dfrac{x}{\beta}\right)^{\alpha\theta}\right]^{-(\alpha+1)/\alpha}, & \text{if } 0<x\\ 0, & \text{if } x\leq 0\end{cases}$$

and

$$F(x)=\begin{cases}\left[\dfrac{\left(\dfrac{x}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\dfrac{x}{\beta}\right)^{\alpha\theta}}\right]^{1/\alpha}, & \text{if } 0<x\\ 0, & \text{if } x\leq 0\end{cases}$$

respectively, where $\alpha>0$, $\beta>0$, and $\theta>0$. Thus, the kappa(3) distribution is characterized by the three parameters α, β, and θ. Parameter β is a scale parameter, whereas parameters α and θ are shape parameters. It appears appropriate at times to set $\theta=1$. In this latter case, parameter α is the only shape parameter and the resulting family of distributions is termed the "two-parameter kappa family of distributions" [or simply the "kappa(2) distribution"].

Procedures to obtain maximum likelihood estimates of the kappa(3) distribution parameters and to perform associated likelihood ratio tests are presented. Also included are numerical examples based on precipitation data collected during a cloud seeding experiment conducted in Florida (Simpson 1972). In addition, specific numerical comparisons using the maximized likelihood function as a goodness-of-fit criterion are made between the gamma, log-normal, and kappa distributions. These latter comparisons are based on precipitation and streamflow data. The precipitation data are associated with weather modification experiments conducted in Colorado and Florida (Mielke et al. 1971, Simpson et al. 1971). The streamflow data illustrate a case in which the kappa(3) distribution reasonably describes a set of data while the kappa(2) distribution is inadequate.

2. MOMENTS AND ORDER STATISTICS

If X is a kappa(3)-distributed random variable, then the rth moment of X about zero is

$$\mu_r'=E\{X^r\}=\beta^r\alpha^{-(\alpha\theta-r)/\alpha\theta}B\left(\frac{\theta+r}{\alpha\theta},\frac{\alpha\theta-r}{\alpha\theta}\right)$$

where

$$B(\delta,\epsilon)=\frac{\Gamma(\delta)\Gamma(\epsilon)}{\Gamma(\delta+\epsilon)}.$$

In particular, μ_r' exists whenever $-\theta<r<\alpha\theta$.

The closed form of the kappa distribution's cumulative distribution function permits ready computation of distributions based on order statistics. For example, let $X^{(k)}$ denote the kth largest order statistic from below in a sample of size n ($k=1, \ldots, n$). Then the density function and cumulative distribution of $X^{(k)}$ are given by

$$f_k(x)=k\binom{n}{k}[F(x)]^{k-1}[1-F(x)]^{n-k}f(x)$$

and

$$F_k(x)=\sum_{j=k}^{n}\binom{n}{j}[F(x)]^{j}[1-F(x)]^{n-j},$$

respectively, where the $f(x)$ and $F(x)$ associated with the kappa(3) distribution are stated explicitly in section 1. Similar computations involving order statistics of either the gamma or log-normal distributions are extremely cumbersome.

3. MAXIMUM LIKELIHOOD ESTIMATORS

Consider a random sample of n observations ($x_1, \ldots, x_n$) from the kappa (3) distribution. The likelihood function associated with this sample is given by

$$L(\alpha, \beta, \theta)=\left(\frac{\alpha\theta}{\beta}\right)^n \prod_{i=1}^{n}\left(\frac{x_i}{\beta}\right)^{\theta-1}\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right]^{-(\alpha+1)/\alpha}.$$

Computational facility is improved by working with the natural logarithm of the likelihood function given by

$$Q=\ln [L(\alpha, \beta, \theta)]=n \ln (\alpha)+n \ln (\theta)-n\theta \ln (\beta)$$
$$+(\theta-1)\sum_{i=1}^{n}\ln (x_i)-\frac{\alpha+1}{\alpha}\sum_{i=1}^{n}\ln\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right].$$

The maximum likelihood estimators of α, β, and θ are those values, say $\hat{\alpha}$, β, and $\hat{\theta}$, which maximize Q (Rao 1965). The Newton-Raphson method for simultaneous equations (Scarborough 1955) is used to iteratively obtain these values. The basis for this method involves truncating all second and higher order terms in the Taylor series expansions of the first-order partial derivatives of Q and setting these truncated expansions equal to zero. The details are given in appendix 1.

For the numerical examples considered, this method converged rapidly and required very little computer time [less than 10s of central processer (CP) time on a Control Data Corporation (CDC) 6400].[1] Appropriate initial values were attained by setting $\theta = 1$ and obtaining the kappa (2) distribution maximum likelihood estimators of α and β (Mielke 1973). The termination criterion for this and subsequent iteration processes is that the absolute difference between the ith and ith + 1 iterative solutions of each parameter is less than 0.0001.

4. EVALUATION PROCEDURES

Initially described is a likelihood ratio test based on the kappa (3) distribution for evaluating possible effects of cloud seeding on precipitation amounts. This test is suitable for experiments based on a random experimental design (Schickedanz and Huff 1971) such as certain Colorado and Florida experiments (Mielke et al. 1971, Simpson et al. 1971).

Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ denote independent random samples of observed precipitation amounts during m nonseeded and n seeded experimental units (e.g., days or clouds), respectively. The likelihood function is given by

$$L(\alpha', \alpha'', \beta', \beta'', \theta', \theta'')=L'(\alpha', \beta', \theta')L''(\alpha'', \beta'', \theta'')$$

where

$$L'(\alpha, \beta, \theta)=\left(\frac{\alpha\theta}{\beta}\right)^m \prod_{i=1}^{m}\left(\frac{x_i}{\beta}\right)^{\theta-1}\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right]^{-(\alpha+1)/\alpha}$$

and

$$L''(\alpha, \beta, \theta)=\left(\frac{\alpha\theta}{\beta}\right)^n \prod_{i=1}^{n}\left(\frac{y_i}{\beta}\right)^{\theta-1}\left[\alpha+\left(\frac{y_i}{\beta}\right)^{\alpha\theta}\right]^{-(\alpha+1)/\alpha}.$$

The natural logarithm of this likelihood function is given by

$$Q=Q'+Q''$$

where

$$Q=\ln [L(\alpha', \alpha'', \beta', \beta'', \theta', \theta'')],$$

$$Q'=\ln [L'(\alpha', \beta', \theta')],$$

and

$$Q''=\ln [L''(\alpha'', \beta'', \theta'')].$$

To investigate a possible scale change induced by seeding, we tested the null parameter space (null hypothesis) given by

$$\omega_1=\{\alpha'=\alpha''=\alpha,\ \beta'=\beta''=\beta,\ \theta'=\theta''=\theta\}$$

against the alternative parameter space (alternative hypothesis) given by

$$\Omega_1=\{\alpha'=\alpha''=\alpha,\ \beta',\ \beta'',\ \theta'=\theta''=\theta\}.$$

If the parameter space is in fact ω_1, then the approximate large sample distribution of the likelihood ratio test statistic given by

$$T_1=2[Q(\hat{\Omega}_1)-Q(\hat{\omega}_1)]$$

is chi-square with 1 degree of freedom. Here, $\hat{\Omega}_1$ and $\hat{\omega}_1$ designate the maximum likelihood estimates of the parameters comprising Ω_1 and ω_1, respectively (Wilks 1962). In particular, the $\hat{\omega}_1$ estimates are obtained from the techniques indicated in section 3 by simply assuming that the $m+n$ observed precipitation amounts in the pooled sample are from the same kappa (3) distribution. However, the $\hat{\Omega}_1$ estimates require an additional application of the Newton-Raphson method, which is presented in appendix 2. This procedure also converged rapidly for the numerical examples that were considered (less than 15 s CP time on a CDC 6400).

[1] Mention of a commercial product does not constitute an endorsement.

If the assumption involving common shape parameters is questioned, the null parameter space given by

$$\omega_2=\{\alpha'=\alpha''=\alpha,\ \beta',\ \beta'',\ \theta'=\theta''=\theta\}$$

can be tested against the alternative parameter space given by

$$\Omega_2=\{\alpha',\ \alpha'',\ \beta',\ \beta'',\ \theta',\ \theta''\}.$$

The appropriate large sample distribution of the likelihood ratio test statistic given by

$$T_2=2[Q(\hat{\Omega}_2)-Q(\hat{\omega}_2)],$$

when the actual parameter space is ω_2, is chi-square with 2 degrees of freedom. In this instance, the $\hat{\omega}_2$ estimates are the same as the previously described $\hat{\Omega}_1$ estimates. Also the $\hat{\Omega}_2$ estimates are obtained from the techniques given in section 3 by obtaining separately the maximum likelihood estimates of the two sets of parameters, $\{\alpha', \beta', \theta'\}$ and $\{\alpha'', \beta'', \theta''\}$, associated with the m nonseeded and n seeded experimental units, respectively.

One might consider, incidentally, whether the use of the kappa(3) distribution [rather than the kappa(2) distribution] results in the improved fit of certain data. For this purpose, the null parameter space given by

$$\omega_3=\{\alpha,\ \beta,\ \theta=1\}$$

is tested against the alternative parameter space given by

$$\Omega_3=\{\alpha,\ \beta,\ \theta\}.$$

If the actual parameter space is ω_3, then the approximate large sample distribution of the likelihood ratio test statistic given by

$$T_3=2[Q(\hat{\Omega}_3)-Q(\hat{\omega}_3)]$$

is chi-square with 1 degree of freedom. For this situation, the $\hat{\omega}_3$ estimates are obtained by using previously described techniques (Mielke 1973), and the $\hat{\Omega}_3$ estimates are determined with techniques given in appendix 1. If the kappa(2) distribution happens to be appropriate, then simple two-sample rank tests based on powers of ranks have optimum properties in their ability to detect scale differences (Mielke 1972, 1973).

5. NUMERICAL EXAMPLES AND COMPARISONS

Application of Evaluation Procedures

To illustrate the evaluation procedures introduced in section 4, we shall consider 52 raw rainfall amounts associated with 26 nonseeded and 26 seeded experimental units of randomized pyrotechnic seeding experiments (Simpson 1972). These experiments were conducted in southern Florida during 1968 and 1970. Among the 26 nonseeded rainfall amounts, a single zero value was replaced by the value 1. The remaining 51 nonseeded and seeded rainfall amount values ranged from 4.1 to 2,745.6 (units are in acre-feet).

TABLE 1.—*Likelihood ratio test statistic values and selected intermediate results*

$\hat{\Omega}_1$:	$\hat{\alpha}=1.384$	$\hat{\beta}'=64.324$	$\hat{\theta}=0.916$
		$\hat{\beta}''=216.511$	
	$Q(\hat{\Omega}_1)=-335.1383$		
$\hat{\omega}_1$:	$\hat{\alpha}=1.463$	$\hat{\beta}=125.935$	$\hat{\theta}=0.819$
	$Q(\hat{\omega}_1)=-338.8461$		
	$T_1=7.4156$		
$\hat{\Omega}_2$:	$\hat{\alpha}'=1.013$	$\hat{\beta}'=54.934$	$\hat{\theta}'=1.085$
	$\hat{\alpha}''=1.871$	$\hat{\beta}''=252.628$	$\hat{\theta}''=0.809$
	$Q(\hat{\Omega}_2)=-334.8745$		
$\hat{\omega}_2$:	$\hat{\alpha}=1.384$	$\hat{\beta}'=64.324$	$\hat{\theta}=0.916$
		$\hat{\beta}''=216.511$	
	$Q(\hat{\omega}_2)=-335.1383$		
	$T_2=0.5276$		
$\hat{\Omega}_3$:	$\hat{\alpha}=1.013$	$\hat{\beta}=54.934$	$\hat{\theta}=1.085$
	$Q(\hat{\Omega}_3)=-153.1878$		
$\hat{\omega}_3$:	$\hat{\alpha}=1.131$	$\hat{\beta}=57.896$	
	$Q(\hat{\omega}_3)=-153.2019$		
	$T_3=0.0282$		

The likelihood ratio test statistics T_1, T_2, and T_3, together with their associated maximum likelihood estimates and maximized likelihood function values, are presented in table 1. While T_1 and T_2 employ the entire set of 52 experimental units, T_3 is based strictly on the 26 nonseeded experimental units.

The approximate p value (probability of having a more extreme test statistic than the realized test statistic under the null hypothesis) associated with the realized value of T_1 is 0.006. In addition, the maximum likelihood estimate of the ratio β''/β' is 3.37 (i.e., an estimated 237 percent increase in rainfall attributed to seeding). These last results merely support previous findings (Simpson et al. 1971). The approximate p value corresponding to the realized value of T_2 is 0.77. This last result suggests that the assumption involving common shape parameters is not unreasonable.

The approximate p value associated with the realized value of T_3 is 0.87. Thus, for the present data, the kappa-(3) distribution offers no improvement of consequence over the kappa(2) distribution. For the present situation, the maximum likelihood estimate of the kappa(2) distribution's shape parameter, $\hat{\alpha}=1.131$, yields a nonparametric power of ranks test (where 1.131 is the power of ranks) that, for large sample sizes, is very efficient in being able to detect small-scale changes (Mielke 1972, 1973). The approximate one-sided p value corresponding to this nonparametric test statistic's realized value is 0.006. Incidentally, the corresponding approximate one-sided p value associated with the realized value of T_1 is 0.003.

TABLE 2.—*Streamflow amounts (1,000 acre-feet) at USGS gaging station number 9-3425 for April 1-August 31 of each year*

1936	192. 48	1954	126. 46
1937	303. 91	1955	128. 58
1938	301. 26	1956	155. 62
1939	135. 87	1957	400. 93
1940	126. 52	1958	248. 57
1941	474. 25	1959	91. 27
1942	297. 17	1960	238. 71
1943	196. 47	1961	140. 76
1944	327. 64	1962	228. 28
1945	261. 34	1963	104. 75
1946	96. 26	1964	125. 29
1947	160. 52	1965	366. 22
1948	314. 60	1966	192. 01
1949	346. 30	1967	149. 74
1950	154. 44	1968	224. 58
1951	111. 16	1969	242. 19
1952	389. 92	1970	151. 25
1953	157. 93		

Empirical Comparisons With Alternative Distributions

The kappa(2) and kappa(3) distributions are compared with the gamma and log-normal distributions for four sets of data. Data set A consists of the raw rainfall amounts associated with the 26 seeded experimental units of the previously mentioned Florida experiments. Data set B consists of the fourth root of the individual raw rainfall amounts associated with the same 26 seeded experimental units (Simpson 1972). Data set C consists of nonzero precipitation amounts associated with 30 specified experimental units of a wintertime orographic cloud seeding experiment conducted in the vicinity of Climax, Colo. (Mielke 1973). Data set D consists of April 1–August 31 accumulated streamflow amounts (units in 1000 acre-feet) for 35 yr (1936–70) at the U.S. Geological Survey (USGS) gaging station number 9–3425 (San Juan River at Pagosa Springs, Colo.). Data set D was extracted from USGS surface water records and is given in table 2.

The maximized likelihood function is both cogent and appealing as a criterion for goodness-of-fit and serves as the goodness-of-fit measure for these comparisions. The maximum likelihood estimators and natural logarithms of the maximized likelihood functions associated with these comparisons are given in table 3. The density functions of the gamma (G) and log-normal (L) distributions associated with the natural logarithms of maximized likelihood functions listed in table 3 have the form

$$f_G(x)=\begin{cases}[\beta^\alpha\Gamma(\alpha)]^{-1}x^{\alpha-1}e^{-x/\beta} & \text{if } 0<x\\ 0 & \text{if } x\leq 0\end{cases}$$

and

$$f_L(x)=\begin{cases}[\alpha x(2\pi)^{1/2}]^{-1}\exp\dfrac{-\left[\ln\left(\dfrac{x}{\beta}\right)\right]^2}{2\alpha^2} & \text{if } 0<x\\ 0 & \text{if } x\leq 0\end{cases}$$

where α and β represent shape and scale parameters, respectively.

TABLE 3.—*Comparisons of kappa (2), kappa (3), gamma and log-normal distributions for four sets of data*

Data set A	
Kappa (2):	Gamma:
$\hat{\alpha}=1.356$	$\hat{\alpha}=0.640$
$\hat{\beta}=216.398$	$\hat{\beta}=691.048$
$Q(\hat{\omega}_3)=-181.8575$	$Q_G(\hat{\alpha},\hat{\beta})=-182.3059$
Kappa (3):	Log-normal:
$\hat{\alpha}=1.871$	$\hat{\alpha}=1.569$
$\hat{\beta}=252.628$	$\hat{\beta}=169.726$
$\hat{\theta}=0.809$	$Q_L(\hat{\alpha},\hat{\beta})=-182.0836$
$Q(\hat{\Omega}_3)=-181.6867$	
Data set B	
Kappa (2):	Gamma:
$\hat{\alpha}=12.395$	$\hat{\alpha}=7.105$
$\hat{\beta}=5.302$	$\hat{\beta}=0.546$
$Q(\hat{\omega}_3)=-51.9326$	$Q_G(\hat{\alpha},\hat{\beta})=-45.3861$
Kappa (3):	Log-normal:
$\hat{\alpha}=1.871$	$\hat{\alpha}=0.392$
$\hat{\beta}=3.987$	$\hat{\beta}=3.609$
$\hat{\theta}=3.237$	$Q_L(\hat{\alpha},\hat{\beta})=-45.9233$
$Q(\hat{\Omega}_3)=-45.5264$	
Data Set C	
Kappa (2):	Gamma:
$\hat{\alpha}=1.8644$	$\hat{\alpha}=0.9687$
$\hat{\beta}=0.0695$	$\hat{\beta}=0.0993$
$Q(\hat{\omega}_3)=40.5817$	$Q_G(\hat{\alpha},\hat{\beta})=40.2601$
Kappa (3):	Log-normal:
$\hat{\alpha}=1.0459$	$\hat{\alpha}=1.1422$
$\hat{\beta}=0.0548$	$\hat{\beta}=0.0529$
$\hat{\theta}=1.4455$	$Q_L(\hat{\alpha},\hat{\beta})=41.6364$
$Q(\hat{\Omega}_3)=40.8103$	
Data set D	
Kappa (2):	Gamma:
$\hat{\alpha}=10.7421$	$\hat{\alpha}=5.2178$
$\hat{\beta}=312.1797$	$\hat{\beta}=41.9621$
$Q(\hat{\omega}_3)=-214.1449$	$Q_G(\hat{\alpha},\hat{\beta})=-207.0155$
Kappa (3):	Log-normal:
$\hat{\alpha}=0.0345$	$\hat{\alpha}=0.4448$
$\hat{\beta}=161.0220$	$\hat{\beta}=198.3373$
$\hat{\theta}=74.7458$	$Q_L(\hat{\alpha},\hat{\beta})=-206.4585$
$Q(\hat{\Omega}_3)=-207.0034$	

By the present criterion, the kappa(3) distribution is not inferior to either the gamma or log-normal distributions for data sets A, B, C, or D. However, for data set B involving the fourth root of the individual raw rainfall amounts, the kappa(2) distribution is unable to account for the transformation's induced symmetrization of this data. Further, the kappa(2) does not provide an adequate fit of data set D. In particular, the p values associated with the realized values of T_3 for data sets A, B, C, and D are 0.56, 0.0003, 0.50, and 0.0001, respectively.

6. CONCLUSIONS

The kappa(3) distribution compares very favorably with the gamma and log-normal distributions in being able to describe natural precipitation data. In addition, the kappa(3) distribution's cumulative distribution function has a closed form. As a result, computations involving distributions based on the kappa(3) distribution's order statistics are relatively convenient. Similar computations involving cumulative distribution functions not in closed form (e.g., the gamma and log-normal distributions) require either extensive tables or numerical integration procedures.

Numerical procedures for obtaining maximum likelihood estimates of the kappa(3) distribution's parameters have been described. Related numerical procedures for determining likelihood ratio test statistics based on the kappa(3) distribution have also been given. These likelihood ratio tests are suitable for evaluating realized data from weather modification experiments based on a random-experimental design. These numerical procedures required very little computing time (less than 15 sCP time on a CDC 6400 for examples presently considered).

APPENDIX 1

Iterative Procedures for Obtaining Kappa(3) Parameter Likelihood Estimators

Let

$$Q_1(i)=\frac{\partial Q}{\partial\alpha}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

$$Q_2(i)=\frac{\partial Q}{\partial\beta}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

$$Q_3(i)=\frac{\partial Q}{\partial\theta}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

$$Q_{11}(i)=\frac{\partial^2 Q}{\partial\alpha^2}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

$$Q_{22}(i)=\frac{\partial^2 Q}{\partial\beta^2}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

$$Q_{33}(i)=\frac{\partial^2 Q}{\partial\theta^2}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

$$Q_{12}(i)=\frac{\partial^2 Q}{\partial\alpha\partial\beta}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

$$Q_{13}(i)=\frac{\partial^2 Q}{\partial\alpha\partial\theta}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i},$$

and

$$Q_{23}(i)=\frac{\partial^2 Q}{\partial\beta\partial\theta}\bigg|_{\alpha=\alpha_i,\ \beta=\beta_i,\ \theta=\theta_i}$$

where

$$Q=\ln[L(\alpha,\beta,\theta)]=n\ln(\alpha)+n\ln(\theta)-n\theta\ln(\beta)+(\theta-1)\sum_{i=1}^{n}\ln(x_i)-\frac{\alpha+1}{\alpha}\sum_{i=1}^{n}\ln\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right].$$

Explicit expressions for the first- and second-order partial derivatives of Q are

$$Q_1=\frac{n}{\alpha}+\frac{1}{\alpha^2}\sum_{i=1}^{n}\ln\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right]-\frac{\alpha+1}{\alpha}\sum_{i=1}^{n}\frac{1+\theta\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}},$$

$$Q_2=-\frac{n\theta}{\beta}+\frac{(\alpha+1)\theta}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}},$$

$$Q_3=\frac{n}{\theta}-n\ln(\beta)+\sum_{i=1}^{n}\ln(x_i)-(\alpha+1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}},$$

$$Q_{11}=-\frac{n}{\alpha^2}-\frac{2}{\alpha^3}\sum_{i=1}^{n}\ln\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right]+\frac{2}{\alpha^2}\sum_{i=1}^{n}\frac{1+\theta\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}-\frac{(\alpha+1)\theta^2}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\left[\ln\left(\frac{x_i}{\beta}\right)\right]^2}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}+\frac{\alpha+1}{\alpha}\sum_{i=1}^{n}\left[\frac{1+\theta\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}\right]^2,$$

$$Q_{22}=\frac{n\theta}{\beta^2}-\frac{(\alpha+1)(\alpha\theta+1)\theta}{\beta^2}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}+\frac{\alpha(\alpha+1)\theta^2}{\beta^2}\sum_{i=1}^{n}\left[\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}\right]^2,$$

$$Q_{33}=-\frac{n}{\theta^2}-\alpha(\alpha+1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\left[\ln\left(\frac{x_i}{\beta}\right)\right]^2}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}+\alpha(\alpha+1)\sum_{i=1}^{n}\left[\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}\right]^2,$$

$$Q_{12}=\frac{\theta}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}+\frac{(\alpha+1)\theta^2}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}$$

$$-\frac{(\alpha+1)\theta}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\left[1+\theta\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)\right]}{\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right]^2},$$

$$Q_{13}=-\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}-(\alpha+1)\theta\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\left[\ln\left(\frac{x_i}{\beta}\right)\right]^2}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}$$

$$+(\alpha+1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\left[1+\theta\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)\right]\ln\left(\frac{x_i}{\beta}\right)}{\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right]^2},$$

and

$$Q_{23}=-\frac{n}{\beta}+\frac{\alpha+1}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}+\frac{\alpha(\alpha+1)\theta}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}}$$

$$-\frac{\alpha(\alpha+1)\theta}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{2\alpha\theta}\ln\left(\frac{x_i}{\beta}\right)}{\left[\alpha+\left(\frac{x_i}{\beta}\right)^{\alpha\theta}\right]^2}.$$

After truncating all second-and higher order terms in the Taylor series expansions of the first-order partial derivatives of Q and setting these truncated expansions equal to zero, the resulting system of equations is given by

$$Q_1(i)+(\alpha_{i+1}-\alpha_i)Q_{11}(i)+(\beta_{i+1}-\beta_i)Q_{12}(i)+(\theta_{i+1}-\theta_i)Q_{13}(i)=0,$$

$$Q_2(i)+(\alpha_{i+1}-\alpha_i)Q_{12}(i)+(\beta_{i+1}-\beta_i)Q_{22}(i)+(\theta_{i+1}-\theta_i)Q_{23}(i)=0,$$

and

$$Q_3(i)+(\alpha_{i+1}-\alpha_i)Q_{13}(i)+(\beta_{i+1}-\beta_i)Q_{23}(i)+(\theta_{i+1}-\theta_i)Q_{33}(i)=0.$$

This system of equations together with appropriate initial values, say α_0, β_0, and θ_0, provides an iterative process that yields a sequence of solutions converging to $\hat{\alpha}$, β, and $\hat{\theta}$. Specific solutions of the ith+1 iterative solutions (α_{i+1}, β_{i+1}, and θ_{i+1}), given the ith iterative solutions (α_i, β_i, and θ_i), are given by

$$\alpha_{i+1}=\alpha_i+\frac{A_i}{D_i},$$

$$\beta_{i+1}=\beta_i+\frac{B_i}{D_i},$$

and

$$\theta_{i+1}=\theta_i+\frac{C_i}{D_i}$$

where

$$A_i=Q_1(i)[Q_{23}^2(i)-Q_{22}(i)Q_{33}(i)]+Q_2(i)[Q_{33}(i)Q_{12}(i)-Q_{13}(i)Q_{23}(i)]+Q_3(i)[Q_{22}(i)Q_{13}(i)-Q_{12}(i)Q_{23}(i)],$$

$$B_i=Q_1(i)[Q_{33}(i)Q_{12}(i)-Q_{13}(i)Q_{23}(i)]+Q_2(i)[Q_{13}^2(i)-Q_{11}(i)Q_{33}(i)]+Q_3(i)[Q_{11}(i)Q_{23}(i)-Q_{12}(i)Q_{13}(i)],$$

$$C_i=Q_1(i)[Q_{22}(i)Q_{13}(i)-Q_{12}(i)Q_{23}(i)]+Q_2(i)[Q_{11}(i)Q_{23}(i)-Q_{12}(i)Q_{13}(i)]+Q_3(i)[Q_{12}^2(i)-Q_{11}(i)Q_{22}(i)],$$

and

$$D_i=Q_{11}(i)Q_{22}(i)Q_{33}(i)+2Q_{12}(i)Q_{13}(i)Q_{23}(i)-Q_{11}(i)Q_{23}^2(i)-Q_{22}(i)Q_{13}^2(i)-Q_{33}(i)Q_{12}^2(i).$$

APPENDIX 2

Iterative Method for Obtaining $\hat{\Omega}_1$ Estimates

For this case,

$$Q=\ln[L(\alpha,\beta',\beta'',\theta)],$$

$$Q'=\ln[L'(\alpha,\beta',\theta)],$$

and

$$Q''=\ln[L''(\alpha,\beta'',\theta)].$$

Also, if the subscripts 1, 2′, 2″, and 3 designate the partial derivatives with respect to α, β', β'', and θ, respectively, the required partial derivatives of Q, expressed in the same iterative notation used in section 3 and appendix 1, are stated as follows:

$$Q_1(i)=Q_1'(i)+Q_1''(i),\qquad Q_{2'2'}(i)=Q'_{2'2'}(i),$$

$$Q_{2'}(i)=Q'_{2'}(i),\qquad Q_{2''2''}(i)=Q''_{2''2''}(i),$$

$$Q_{2''}(i)=Q''_{2''}(i),\qquad Q_{12'}(i)=Q'_{12'}(i),$$

$$Q_3(i)=Q_3'(i)+Q_3''(i),\qquad Q_{12''}(i)=Q''_{12''}(i),$$

$$Q_{11}(i)=Q_{11}'(i)+Q_{11}''(i),\qquad Q_{2'2''}(i)=0,$$

$$Q_{33}(i)=Q_{33}'(i)+Q_{33}''(i),\qquad Q_{2'3}(i)=Q'_{2'3}(i),\quad\text{and}$$

$$Q_{13}(i)=Q_{13}'(i)+Q_{13}''(i),\qquad Q_{2''3}(i)=Q''_{2''3}(i).$$

Since Q' and Q'' are essentially identical to the Q defined in section 3, explicit expressions for these first- and second-order partial derivatives of Q are easily obtained from appendix 1. For this case, specific solutions of the ith+1 iterative solutions (α_{i+1}, β'_{i+1}, β''_{i+1}, and θ_{i+1}) given the ith iterative solutions (α_i, β'_i, β''_i, and θ_i) are given by

$$\alpha_{i+1}=\alpha_i-[Q_1(i)C_{11}(i)+Q_{2'}(i)C_{12'}(i)+Q_{2''}(i)C_{12''}(i)+Q_3(i)C_{13}(i)],$$

$$\beta'_{i+1}=\beta'_i-[Q_1(i)C_{12'}(i)+Q_{2'}(i)C_{2'2'}(i)+Q_{2''}(i)C_{2'2''}(i)+Q_3(i)C_{2'3}(i)],$$

$$\beta''_{i+1}=\beta''_i-[Q_1(i)C_{12''}(i)+Q_{2'}(i)C_{2'2''}(i)+Q_{2''}(i)C_{2''2''}(i)+Q_3(i)C_{2''3}(i)],$$

and

$$\theta_{i+1}=\theta_i-[Q_1(i)C_{13}(i)+Q_{2'}(i)C_{2'3}(i)+Q_{2''}(i)C_{2''3}(i)+Q_3(i)C_{33}(i)]$$

where

$$C_{11}(i)=[Q_{2'2'}(i)Q_{2''2''}(i)Q_{33}(i)-Q_{2'3}^2(i)Q_{2''2''}(i)-Q_{2''3}^2(i)Q_{2'2'}(i)]\Delta_i^{-1},$$

$$C_{12'}(i)=[Q_{12'}(i)Q^2_{2''3}(i)+Q_{13}(i)Q_{2'3}(i)Q_{2''2''}(i) - Q_{12'}(i)Q_{2''2''}(i)Q_{33}(i) - Q_{12''}(i)Q_{2'3}(i)Q_{2''3}(i)]\Delta_i^{-1},$$

$$C_{12''}(i)=[Q_{12''}(i)Q^2_{2'3}(i)+Q_{13}(i)Q_{2''3}(i)Q_{2'2'}(i) - Q_{12''}(i)Q_{2'2'}(i)Q_{33}(i) - Q_{12'}(i)Q_{2''3}(i)Q_{2'3}(i)]\Delta_i^{-1},$$

$$C_{13}(i)=[Q_{12'}(i)Q_{2'3}(i)Q_{2''2''}(i)+Q_{12''}(i)Q_{2''3}(i)Q_{2'2'}(i) - Q_{13}(i)Q_{2'2'}(i)Q_{2''2''}(i)]\Delta_i^{-1},$$

$$C_{2'2'}(i)=[Q_{11}(i)Q_{2''2''}(i)Q_{33}(i)+2Q_{12''}(i)Q_{13}(i)Q_{2''3}(i) - Q^2_{12''}(i)Q_{33}(i)-Q^2_{13}(i)Q_{2''2''}(i) - Q^2_{2''3}(i)Q_{11}(i)]\Delta_i^{-1},$$

$$C_{2'2''}(i)=[Q_{12'}(i)Q_{12''}(i)Q_{33}(i)+Q_{2'3}(i)Q_{2''3}(i)\,Q_{11}(i) - Q_{12'}(i)Q_{13}(i)Q_{2''3}(i) - Q_{12''}(i)Q_{13}(i)Q_{2'3}(i)]\Delta_i^{-1},$$

$$C_{2'3}(i)=[Q^2_{12''}(i)Q_{2'3}(i)+Q_{12'}(i)Q_{13}(i)Q_{2''2''}(i) - Q_{11}(i)Q_{2''2''}(i)Q_{2'3}(i) - Q_{12'}(i)Q_{12''}(i)Q_{2''3}(i)]\Delta_i^{-1},$$

$$C_{2''2''}(i)=[Q_{11}(i)Q_{2'2'}(i)Q_{33}(i)+2Q_{12'}(i)Q_{13}(i)Q_{2'3}(i) - Q^2_{12'}(i)Q_{33}(i)-Q^2_{13}(i)Q_{2'2'}(i) - Q^2_{2'3}(i)Q_{11}(i)]\Delta_i^{-1},$$

$$C_{2''3}(i)=[Q^2_{12'}(i)Q_{2''3}(i)+Q_{12''}(i)Q_{13}(i)Q_{2'2'}(i) - Q_{11}(i)Q_{2'2'}(i)Q_{2''3}(i) - Q_{12'}(i)Q_{12''}(i)Q_{2'3}(i)]\Delta_i^{-1},$$

$$C_{33}(i)=[Q_{11}(i)Q_{2'2'}(i)Q_{2''2''}(i)-Q^2_{12'}(i)Q_{2''2''}(i) - Q^2_{12''}(i)Q_{2'2'}(i)]\Delta_i^{-1},$$

and

$$\begin{aligned}\Delta_i=&Q_{11}(i)Q_{2'2'}(i)Q_{2''2''}(i)Q_{33}(i)+Q^2_{12'}(i)Q^2_{2''3}(i)\\ &+Q^2_{12''}(i)Q^2_{2'3}(i)+2Q_{12'}(i)Q_{13}(i)Q_{2'3}(i)Q_{2''2''}(i)\\ &+2Q_{12''}(i)Q_{13}(i)Q_{2''3}(i)Q_{2'2'}(i)\\ &-Q^2_{12'}(i)Q_{2''2''}(i)Q_{33}(i)-Q^2_{12''}(i)Q_{2'2'}(i)Q_{33}(i)\\ &-Q^2_{13}(i)Q_{2'2'}(i)Q_{2''2''}(i)-Q^2_{2'3}(i)Q_{11}(i)Q_{2''2''}(i)\\ &-Q^2_{2''3}(i)Q_{11}(i)Q_{2'2'}(i)\\ &-2Q_{12'}(i)Q_{12''}(i)Q_{2'3}(i)Q_{2''3}(i).\end{aligned}$$

These solutions, together with appropriate initial values (α_0, β_0', β_0'', and θ_0), provide a sequence of solutions converging to the $\hat{\Omega}_1$ estimates.

ACKNOWLEDGMENTS

This study was jointly supported by the National Science Foundation, Division of Environmental Systems and Resources, Weather Modification Program, under Grant GI-31460, Amend. No. 1 and a cooperative agreement with the Rocky Mountain Forest and Range Experiment Station and the Northern Forest Fire Laboratory, USDA Forest Service. The authors appreciate helpful comments by the reviewers.

REFERENCES

Mielke, Paul W., Jr., "Asymptotic Behavior of Two-Sample Tests Based on Powers of Ranks for Detecting Scale and Location Alternatives," *Journal of the American Statistical Association*, Vol. 67, No. 340, Dec. **1972**, pp. 850–854.

Mielke, Paul W., Jr., "Another Family of Distributions for Describing and Analyzing Precipitation Data," *Journal of Applied Meteorology*, Vol. 12, No. 2, Mar. **1973**, pp. 275–280.

Mielke, Paul W., Jr., Grant, Lewis O., and Chappell, Charles F., "An Independent Replication of the Climax Wintertime Orographic Cloud Seeding Experiment," *Journal of Applied Meteorology*, Vol. 10, No. 6, Dec. **1971**, pp. 1198–1212.

Neyman, Jerzy, and Scott, Elizabeth L., "Asymptotically Optimum Tests of Composite Hypotheses For Randomized Experiments with Noncontrolled Predictor Variables," *Journal of the American Statistical Association*, Vol. 60, No. 311, Sept. **1965**, pp. 699–721.

Rao, C. Radhakrishna, *Linear Statistical Inference and its Applications*, John Wiley & Sons, Inc., New York, N.Y., **1965**, 522 pp. (See Chapter 5, pp. 289–291).

Scarborough, James B., *Numerical Mathematical Analysis* (3d Ed.), The Johns Hopkins University Press, Baltimore, Md., **1955**, 554 pp. (See Chapter 9, pp. 203–206).

Schickedanz, Paul T., and Huff, Floyd A., "The Design and Evaluation of Rainfall Modification Experiments," *Journal of Applied Meteorology*, Vol. 10, No. 3, June **1971**, pp. 502–514.

Simpson, Joanne, "Use of the Gamma Distribution in Single-Cloud Rainfall Analysis," *Monthly Weather Review*, Vol. 100, No. 4, Apr. **1972**, pp. 309–312.

Simpson, Joanne, Woodley, William L., Miller, Alan H., and Cotton, Gerald F., "Precipitation Results of Two Randomized Pyrotechnic Cumulus Seeding Experiments," *Journal of Applied Meteorology*, Vol. 10, No. 3, June **1971**, pp. 526–544.

Thom, Herbert C. S., "A Statistical Method of Evaluating Augmentation of Precipitation by Cloud Seeding," *Final Report of the Advisory Committee on Weather Control*, Vol. 2, U.S. Government Printing Office, Washington, D.C., **1957**, pp. 5–25.

Wilks, Samuel S., *Mathematical Statistics*, John Wiley & Sons, Inc., New York, N.Y., **1962**, 644 pp. (See Chapter 13, pp. 402–422).

[*Received March 12, 1973; revised August 2, 1973*]

UDC 551.558.21:551.507.362.2(084.1)"1973.05.22"

PICTURE OF THE MONTH
Gravity Waves Following Severe Thunderstorms

CARL O. ERICKSON and LINWOOD F. WHITNEY, JR.—*Meteorological Satellite Laboratory, National Environmental Satellite Service, NOAA, Hillcrest Heights, Md.*

Visible and infrared pictures (figs. 1, 2) from the Very High Resolution Radiometer (VHRR) aboard the NOAA 2 satellite both show the eastern half of the United States after a night of severe thunderstorm activity over the Southern Great Plains. Picture times are identical and are approximately 1010 CDT (1510 GMT) on May 22, 1973. A partial grid has been superimposed on the infrared view.

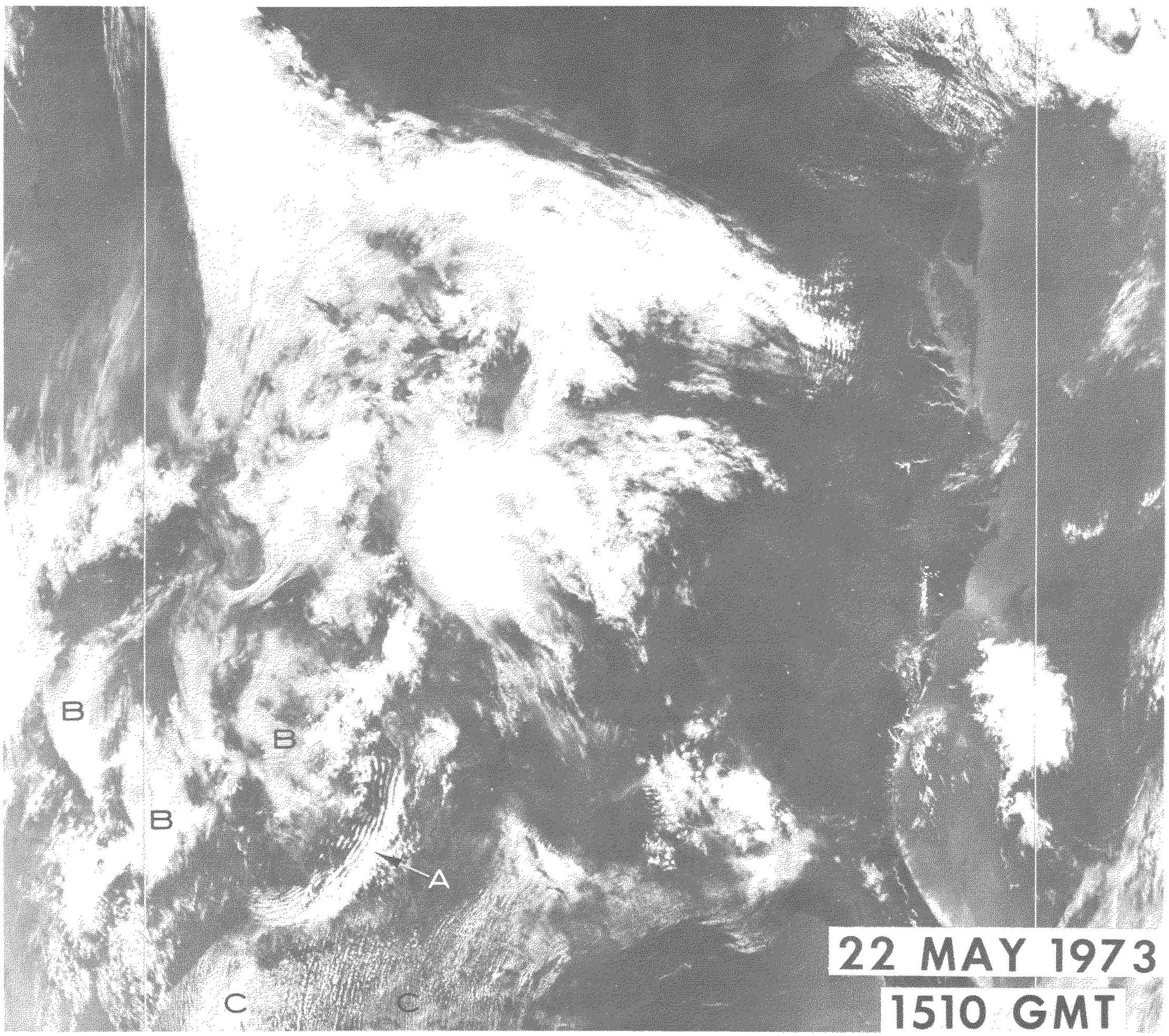

FIGURE 1.—View of the Eastern United States as seen through the visible channel of the Very High Resolution Radiometer aboard the NOAA #2 satellite (0.6–0.7μm). Picture time is approximately 1510 GMT, May 22, 1973.

Of particular interest is the wave-cloud formation at A, which extends from east-central Arkansas to northeastern Texas. Figure 3 is an enlargement of the visible view of the area. This cloud formation at A is believed to represent southeastward-moving gravity waves initiated by the earlier violent convection to the north and northwest. During the night of May 21 and the early morning of May 22, numerous severe thunderstorms were reported from the five-state area of Texas, Arkansas, Oklahoma, Kansas, and Missouri, a few storms persisting until after sunrise. Tornadoes also occurred over Kansas, Missouri, Iowa, and Arkansas. Cloud debris associated with the earlier convection is indicated by B.

During and following that thunderstorm activity, a number of surface observing stations in the area reported either the passage of a pressure jump or the occurrence of markedly unsteady surface pressure. Over northeastern Texas and Arkansas, there were also a few reports indicating the existence of a traveling roll cloud and windshift line during the 2–4 hr preceding picture time. Figure 4 gives the surface analysis for 1500 GMT—10 minutes prior to picture time. The analysis shows that a traveling surface disturbance continued to exist in the form of a pressure surge and meso-High. The trough line preceding the meso-High is nearly coincident with the satellite-viewed wave clouds.

Most of the wave-cloud formation appears to be at middle tropospheric levels. Both the relative brightness of the formation in the infrared picture of figure 2 and the conventional meteorological data in the area indicate

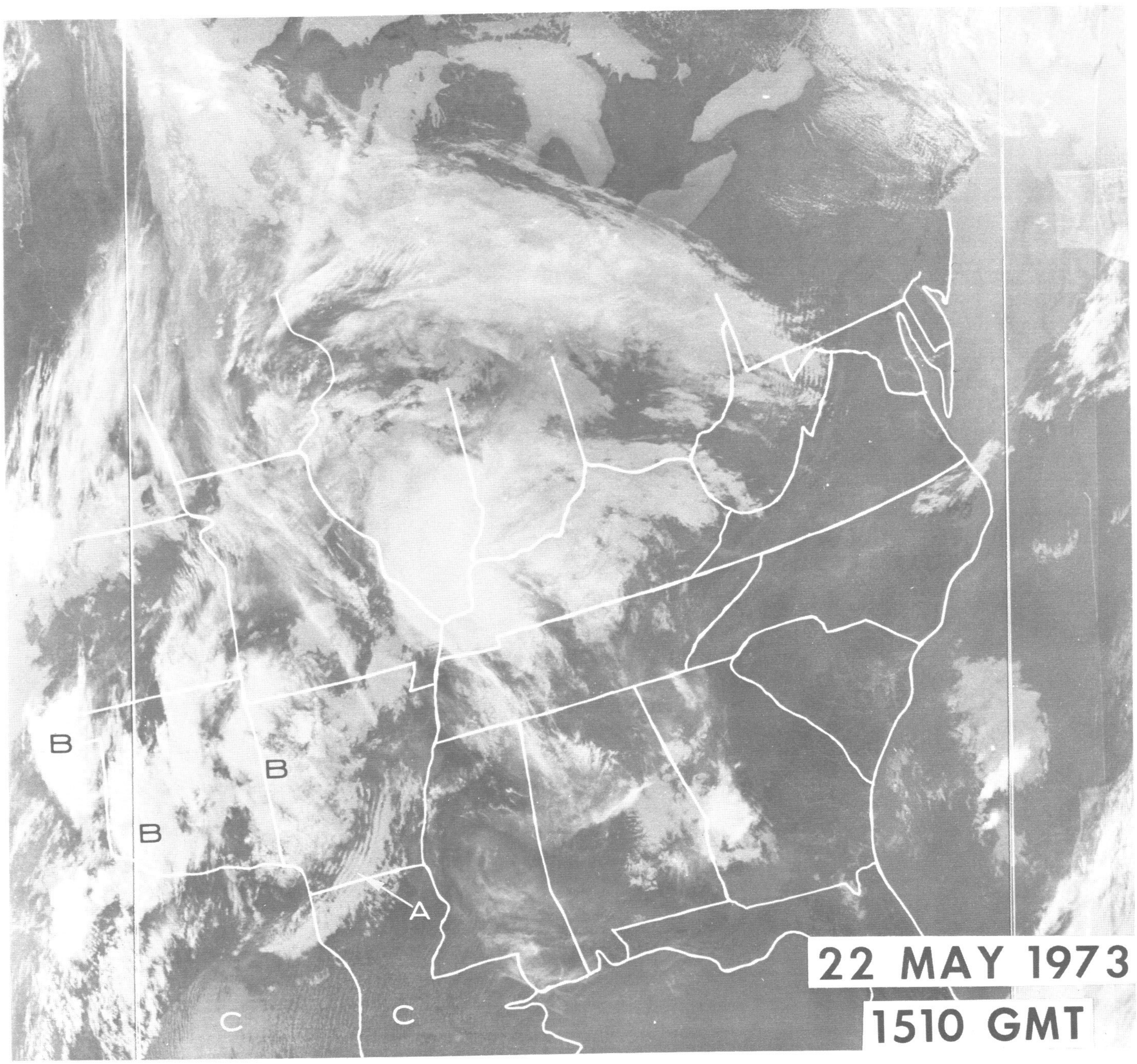

FIGURE 2.—Same as figure 1, an infrared view through the water vapor window channel (10.5–12.5 μm).

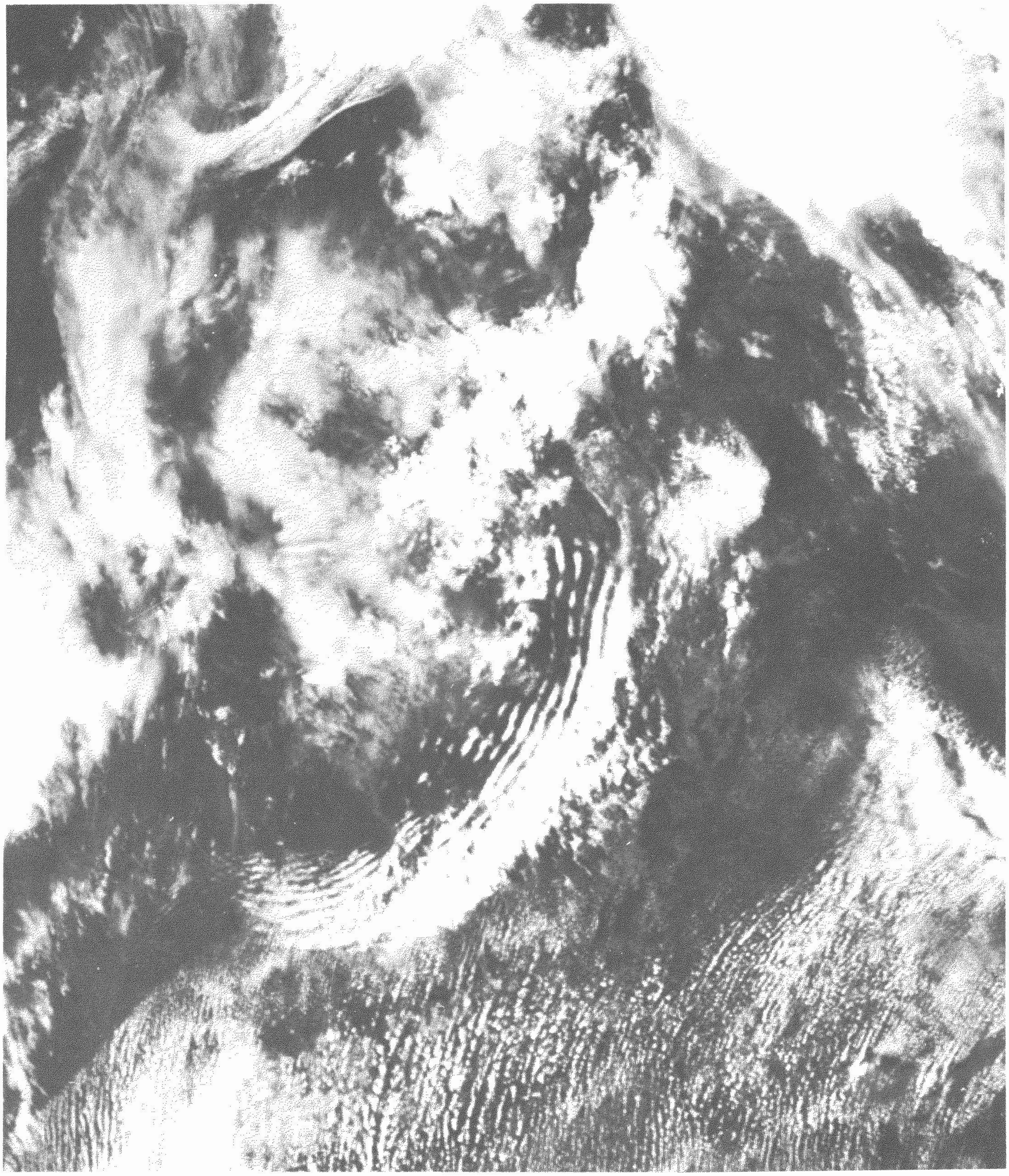

FIGURE 3.—Enlarged view of figure 1 showing wave-cloud area.

an altitude not far from 550 mb. However, the dark gray extension of this wave pattern in northeast Texas (apparent in the original version of fig. 2 but possibly not in the printed reproduction) shows that the low-level stratocumulus is also affected. Thus, the waves appear to exist through a deep layer of the troposphere.

South of the wave clouds, over East Texas, Louisiana, and Southern Mississippi, many small low-level cumuli

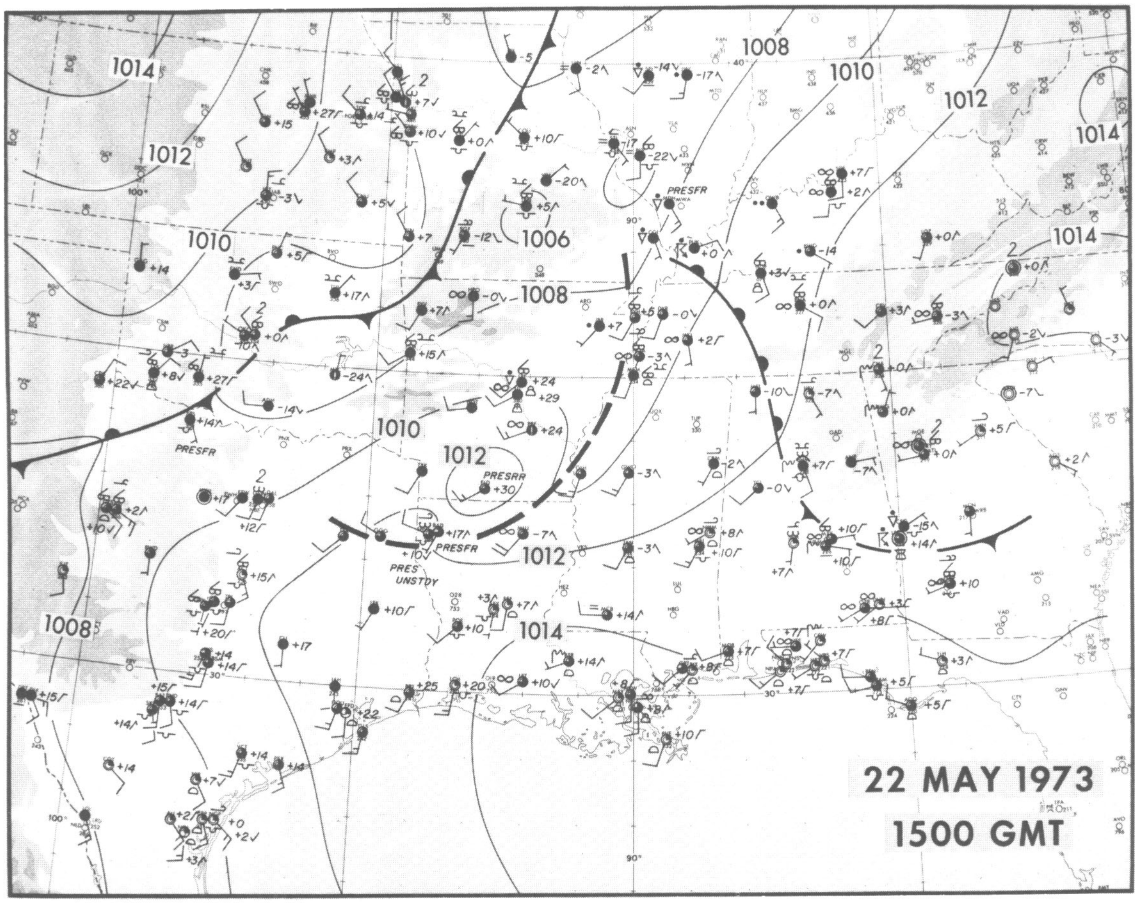

FIGURE 4.—Surface analysis for 1500 GMT, May 22, 1973.

are forming in the morning heating of the tropical air (area C). This area was unaffected by earlier nighttime precipitation. These cumulus clouds are at a much lower elevation than most of the wave clouds immediately to the north, therefore they appear less bright (warmer) in the infrared picture.

Note that the infrared view of figure 2 shows the waters of the Great Lakes (near top of picture) and of the ocean along the Middle Atlantic Coast to be much cooler (brighter) than the adjacent land. Also, the coastal waters are quite bright in the visible (fig. 1) because of sun glint observed by the eastward-looking scans.